Eur. Phys. J. D **1**, 93–95 (1998)

THE EUROPEAN PHYSICAL JOURNAL D

Canonical dressing in the multimode two-quantum Jaynes-Cummings model

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Received: 5 July 1997 / Revised: 3 November 1997 / Accepted: 14 November 1997

Abstract. Based upon the hidden Lie SU(1,1) symmetry, we have constructed the unitary decoupling transformation which diagonalizes the multimode two-quantum Jaynes-Cummings model and provides us with an extremely convenient basis to gain a deeper understanding of the dressing processes present in the matter-field interaction. This canonical transformation approach is very simple and can be easily extended to other generalized Jaynes-Cummings models.

PACS. 03.65.Fd Algebraic methods – 32.80.Pj Optical cooling of atoms; trapping – 42.50.Vk Mechanical effects of light on atoms, molecules, electrons, and ions

For the past three decades the Jaynes-Cummings (JC) model has been playing a very significant role in our understanding of the interaction between radiation and matter in quantum optics [1]. This rather simplified but nontrivial model idealizes the real situation by simply concentrating on the near resonant linear coupling between a single two-level atomic system and a quantized radiation mode ($\hbar = 1$):

$$H = \omega_0 S_z + \omega a^\dagger a + \epsilon \left(a^\dagger S_- + a S_+ \right), \qquad (1)$$

where the radiation mode of frequency ω is described by the bosonic operators a and $a^\dagger$, the two atomic levels separated by an energy difference ω_0 are represented by the spin-half operators S_z and $S_\pm$, and the atom-field coupling strength is measured by the positive parameter ϵ. Despite its simplicity, the JC model is of great significance because recent technological advances have enabled us to experimentally realize this rather idealized model [2–5] and to verify some of the theoretical predictions. Furthermore, it has been recently shown that a single trapped and laser-irradiated ion exhibits a strongly nonlinear JC dynamics [6]. Here the quantized center-of-mass motion of the ion in the harmonic trap potential plays the role of a boson mode, which is coupled *via* the laser to the internal (electronic) degrees of freedom. Meanwhile this prediction has been confirmed experimentally [7] and modifications due to micromotion have been studied [8]. In constrast to the JC model in quantum optics, the strength of the laser-induced vibronic coupling can be easily controlled by varying the intensity of the applied laser, and the typical coupling strength ϵ/ω is much stronger, approximately $10^{-2} \sim 10^{-3}$ [7]. Stimulated by the success of the JC model, more and more people have paid special attention to extending and generalizing the model in order to explore new quantum effects [9]. One possible generalization is the multimode two-quantum JC model:

$$H = \omega_0 S_z + \sum_{\mathbf{k}} \omega a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \sum_{\mathbf{k},\mathbf{q}} \epsilon_{\mathbf{k},\mathbf{q}} a_{\mathbf{k}}^\dagger a_{\mathbf{q}}^\dagger S_- + \epsilon_{\mathbf{k},\mathbf{q}}^* a_{\mathbf{k}} a_{\mathbf{q}} S_+, \qquad (2)$$

where all the boson modes are degenerate. Such a generalization is of considerable interest because of its relevance to the study of the coupling between a single atomic two-level system and the boson field with the atom making two-quantum transitions [10–13]. This generalized model also allows us to investigate the multimode squeezing and the effect of intermode correlation on the atomic transitions.

As in the original JC model, the exact eigenstates of the system of this generalized case can *in principle*, though not feasible, be obtained by diagonalizing the secular matrix of the total many-body Hamiltonian [1]. These eigenstates identify a new physical unit, known as a *dressed atom*, representing the combined atom-field system in the sense that its spectrum coincides with the energy spectrum of the generalized JC Hamiltonian and its stationary states, called *dressed states*, are precisely the eigenstates of the coupled system. This algebraic approach of the atomic dressing, however, does not seem to offer an incisive tool to catch the dressing meaning and its role. Accordingly, in this paper we would like to present an alternative approach based upon constructing an appropriate unitary transformation which accomplishes the exact canonical dressing of a two-level atomic system by the boson field. This canonical approach has been previously applied to the original JC model and its multimode extension,

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and has proved itself to be much more effective than the algebraic approach in elucidating the physical origin of the dressing processes [14,15]. Furthermore, the advantages of having an explicit unitary operator which performs the required dressing of the two-level atomic system are obvious since any quantum mechanical computation can be performed by directly transforming the operators appearing in the problem at hand without going through the wavefunctions, which are of a very complicated structure, especially in the multimode case.

To begin with, let us introduce the operators K_+, K_- and K_0 [16]:

$$\begin{aligned} K_+ &= \frac{1}{2}\sum_{\mathbf{k},\mathbf{q}} \alpha_{\mathbf{k},\mathbf{q}} a^\dagger_{\mathbf{k}} a^\dagger_{\mathbf{q}}, \\ K_- &= K_+^\dagger = \frac{1}{2}\sum_{\mathbf{k},\mathbf{q}} \alpha^*_{\mathbf{k},\mathbf{q}} a_{\mathbf{k}} a_{\mathbf{q}}, \\ K_0 &= \frac{1}{4}\sum_{\mathbf{k}} (2a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + 1) \end{aligned} \tag{3}$$

where $\alpha_{\mathbf{k},\mathbf{q}}$ is a symmetric unitary matrix. These three operators represent the three generators of the multimode bosonic realization of the closed Lie algebra SU(1,1), which is defined by the commutation relations [17]:

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0. \tag{4}$$

The matrix $\alpha_{\mathbf{k},\mathbf{q}}$ may take the special form

$$\alpha_{\mathbf{k},\mathbf{q}} = \exp(i\phi) - \eta_{\mathbf{k}}\eta_{\mathbf{q}} \tag{5}$$

with $\eta_{\mathbf{k}}$ being real and $0 \le \phi \le \pi/2$. The unitarity condition then requires

$$\sum_{\mathbf{k}} u^2_{\mathbf{k}} = \sum_{\mathbf{k}} \left[\frac{\eta_{\mathbf{k}}}{\sqrt{2\cos(\phi)}}\right]^2 = 1, \tag{6}$$

implying that the multi-dimensional vector $\mathbf{u}$ is just the unit radial vector in the multi-dimensional space. In other words, each point on the multi-dimensional unit sphere corresponds to a possible representation of the symmetric unitary matrix $\alpha_{\mathbf{k},\mathbf{q}}$. The corresponding Casimir operator C is given by [17]

$$C = K_0^2 - \frac{1}{2}\left(K_+K_- + K_-K_+\right), \tag{7}$$

which has the eigenvalue $k\,(k-1)$ for a Unitary Irreducible Representation (UIR). The parameter k is the so-called Bargmann index. For the UIR known as the positive discrete series $\mathcal{D}^+(k)$, the states $|m,k\rangle$ diagonalize the compact operator K_0 [16]:

$$K_0|m,k\rangle = (m+k)\,|m,k\rangle, \tag{8}$$

for $k = (M+2r)/4$ and $m = 0,1,2,...$ Here M is the total number of boson modes and $r = 0,1,2,...$ (Note that in the single-mode bosonic realization of the SU(1,1) Lie algebra k can be equal to 1/4 or 3/4. For $k = 1/4$ we obtain the even-parity states of the bosonic mode whereas $k = 3/4$ gives the odd-parity states. The vacuum state for the bosonic mode is apparently the state $|0, 1/4\rangle$). The operators K_+ and K_- are Hermitian conjugates of each other and act as raising and lowering operators respectively within $\mathcal{D}^+(k)$, *i.e.*

$$\begin{aligned} K_+|m,k\rangle &= \sqrt{(m+1)\,(m+2k)}\,|m+1,k\rangle \\ K_-|m,k\rangle &= \sqrt{m\,(m+2k-1)}\,|m-1,k\rangle. \end{aligned} \tag{9}$$

The corresponding SU(1,1) generalized coherent states $|z;k\rangle$ are defined as

$$\begin{aligned} |z;k\rangle &= \exp\left(zK_+ - z^*K_-\right)|0,k\rangle \\ &= \exp\left[\frac{1}{2}\left(za^{\dagger 2} - z^*a^2\right)\right]|0,k\rangle. \end{aligned} \tag{10}$$

For $k = 1/4$, the coherent state $|z; k = 1/4\rangle$ is simply the well-known single-mode squeezed vacuum state with the squeezing parameter z.

Provided that $\epsilon_{\mathbf{k},\mathbf{q}} = \epsilon\alpha_{\mathbf{k},\mathbf{q}}$ with ϵ being a real constant, we may rewrite the Hamiltonian in equation (2), in terms of the SU(1,1) generators, as

$$H = H_0 + V \tag{11}$$

where

$$\begin{aligned} H_0 &= \omega_0 S_z + 2\omega\left(K_0 - \frac{M}{4}\right) \\ V &= 2\epsilon\left(K_+S_- + K_-S_+\right). \end{aligned} \tag{12}$$

It is not difficult to see that for a two-mode case with $\alpha_{\mathbf{k},\mathbf{q}} = 1 - \delta_{\mathbf{k},\mathbf{q}}$, the Hamiltonian is reduced to that of the familiar two-mode two-quantum JC model, namely [9]

$$\begin{aligned} H_{2-\text{mode}} =& \omega_0 S_z + \omega\left(a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + a^\dagger_{\mathbf{q}}a_{\mathbf{q}}\right) \\ &+ \epsilon\left(a^\dagger_{\mathbf{k}}a^\dagger_{\mathbf{q}}S_- + a_{\mathbf{k}}a_{\mathbf{q}}S_+\right). \end{aligned} \tag{13}$$

Recognizing that the operator of the total number of excitations

$$\mathcal{N} = \left(K_0 - \frac{M}{4}\right) + S_z + \frac{1}{2}, \tag{14}$$

whose eigenvalues are non-negative definite, commutes with all operators conserving the total number of excitations, *e.g.* $[\mathcal{N}, H_0] = [\mathcal{N}, V] = 0$, we can apply the approach in reference [14] and [15], and can easily show that the Hamiltonian H can be diagonalized by the dressing unitary operator

$$T = \exp\left\{\gamma\left(K_+S_- - K_-S_+\right)\right\} \tag{15}$$

where

$$\begin{aligned} \gamma &= \frac{\theta}{2\beta}, \quad \tan(\theta) = -\frac{4\epsilon\beta}{2\omega - \omega_0} \\ \beta &= \sqrt{\left(\mathcal{N} + \frac{M-4}{4}\right)\left(\mathcal{N} + \frac{M}{4}\right) - C}. \end{aligned} \tag{16}$$

The transformed Hamiltonian $\tilde{H} \equiv T^{\dagger}HT$ takes the diagonal form

$$\tilde{H} = \left(2\omega + \frac{\Delta}{\gamma}\right) S_z + 2\omega\left(K_0 - \frac{M}{4}\right), \quad (17)$$

where $\Delta = \sqrt{(2\epsilon\theta)^2 + \gamma^2 (2\omega - \omega_0)^2}$. The eigenstates of $\tilde{H}$ are simply given by $|m,k,\sigma\rangle \equiv |m,k\rangle|\sigma\rangle$ for $\sigma =\uparrow$ or $\downarrow$, $k = (M+2r)/4$ and $m = 0,1,2,...$ Here M is the total number of boson modes and $r = 0,1,2,...$ The corresponding eigenenergies can be straightforwardly found in the form

$$E_{k,m,\downarrow} = 2\omega\left(m + \frac{r-1}{2}\right) - \frac{\Delta}{2\gamma}$$
$$E_{k,m,\uparrow} = 2\omega\left(m + \frac{r+1}{2}\right) + \frac{\Delta}{2\gamma}. \quad (18)$$

The spin-dependent part in equation (17) can be interpreted as the dressed atom Hamiltonian, and $\tilde{\omega}_0 \equiv 2\omega + \Delta/\gamma$ represents the renormalized energy difference between the two atomic levels in comparison with H_0 in equation (12). It should be noted that since all the boson modes are degenerate, each eigenstate $|m,k\rangle$ is multiply degenerate. For instance, the state $|m=0,k=(M+2)/4\rangle$ is M-fold degenerate and can be realized by each of the simple product states: $|1\rangle_{\mathbf{q}} \prod_{\mathbf{k}\neq\mathbf{q}} |0\rangle_{\mathbf{k}}$, with $\mathbf{q}$ denoting any one of the M boson modes. Furthermore, it is interesting to note that in the two-mode case described by the Hamiltonian in equation (13), there exists another constant of motion, namely $a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}} - a^{\dagger}_{\mathbf{q}}a_{\mathbf{q}}$, which helps distinguish the degenerate states associated with the state $|m,k\rangle$ by providing an additional quantum number.

In summary, based upon the hidden Lie SU(1,1) symmetry, we have constructed the unitary decoupling transformation which diagonalizes the multimode two-quantum Jaynes-Cummings model and provides us with an extremely convenient basis to gain a deeper understanding of the dressing processes present in the matter-field interaction. As pointed out above, a peculiar aspect of this canonical transformation approach is represented by the possibility to evaluate explicit expressions for the dressed operators in terms of the bare ones. For instance, the dressed version of the bare spin operator S_z is given by $\tilde{S_z} \equiv T^{-1}S_zT = \cos(\theta)S_z - (4\epsilon\beta)^{-1}\sin(\theta)V$. It is obvious that this circumstance opens, in principle, the possibility to achieve a much more physically transparent interpretation of the dressed operators than in the algebraic approach. Furthermore, this canonical transformation approach is very simple and can be easily extended to other generalized JC models. Finally, the model studied in this paper should be of interest in experimental situations where suitable time for coherent evolution between a radiation centre and a few degenerate modes of confined radiation is available. These interesting situations can be realized by many current experimental developments such as microcavities in semiconductors [18] and photonic band gap structures [19].

This work is partially supported by the Direct Grant for Research from the Research Grants Council of the Hong Kong Government.

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